Nonlinear Profit-and-Loss-Sharing Contracting versus Equity in Entrepreneurial Finance: Risk Sharing and Managerial Incentives in a Principal-Agent Relationship When the Agent Is Risk-Neutral

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ABSTRACT

An entrepreneur raises capital to finance their business and extends the effort to facilitate its success. If financing is possible solely with risk-free debt capital, the entrepreneur's effort will be first-best since they will not share the outcome of their business with an outside financier. However, risk-free debt is usually unavailable due to the significant risk involved in starting any business. Therefore, other financing arrangements involving risk sharing have to be considered. Such arrangements will result in a moral hazard problem and dilute the entrepreneur's managerial incentives to exert effort. Financing with equity capital from an outside partner will result in a second-best level of effort, and there will be no advantage from risk sharing if the entrepreneur is risk-neutral. Since risk-free debt financing is not always possible and financing with equity capital is not optimal for a risk-neutral entrepreneur, a third alternative is considered in this paper. It is shown that a nonlinear profit-and-loss-sharing arrangement can be structured so that the incentives of the risk-neutral entrepreneur to extend effort are not diminished. As such, the nonlinear PLS contract considered in this paper, which is based on Islamic profit-and-loss-sharing contracts, is a feasible alternative to risk-free debt financing.

Keywords: *Entrepreneurial Finance, Nonlinear Contracts, Profit and Loss Sharing, Risk Sharing, Moral Hazard Problem*

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An entrepreneur with insufficient funds to finance an investment opportunity (i.e., business, project, start-up, etc.) needs outside capital in the form of debt or equity. Hybrid financing, combining the features of debt and equity capital, could also be considered. Financing with equity or hybrid capital provided by a financier involves risk sharing. These methods of financing lead to a principal-agent relationship, with the entrepreneur as the agent and the financier as the principal, resulting in diminished managerial incentives for the entrepreneur to make an effort to facilitate the success of the business since they have to share the outcome with the outside financier. If the entrepreneur can finance their business with risk-free debt capital, which is typically unavailable due to the inherent risks associated with starting a business, their effort is first-best. Risk sharing with an equity partner, on the other hand, results in the second-best effort of the entrepreneur, and without the direct benefits of risk sharing, financing with equity is inferior to risk-free debt financing. The focus of this paper is hybrid financing with the aim of finding an optimal contractual structure that can eliminate (or at least alleviate to some extent) the adverse effects of the moral hazard problem.

Hybrid financial instruments exhibit characteristics of both debt and equity capital. Preferred stock paying dividends at a fixed rate can be considered an example of hybrid capital. Participating preferred stock may also pay out participating dividends, which are linked to the success of a company, in addition to the fixed dividends. Participating loans, on the other hand, entail payments that are dependent on the company's success, as opposed to fixed interest payments. The creditors of participation loans are entitled to a share of the sales revenue or profit; however, they do not share the losses the company might incur. In the case of financing with a participation loan, a fixed payment can also be agreed upon (OECD, 2015). Closer to equity financing are silent partnerships, which can be differentiated into two forms. A typical silent partner who invests in a project participates in profit sharing while an atypical silent partner shares both the profits and the losses with the entrepreneur. The atypical silent partner has more rights as a co-entrepreneur than a typical silent partner for taking on a higher level of risk (OECD, 2015). The remuneration resulting from hybrid financial instruments usually involves a linear sharing of profits. Some might also entail loss sharing or a fixed payment.

Hybrid financing is particularly relevant in project financing and for Small and Medium-Sized Enterprises (SMEs). Examples of hybrid financial instruments can also be found in Islamic finance[1], in the form of near-equity methods of financing that entail nonlinear profit and loss sharing, where the profits and losses resulting from the project are shared between the parties based on different shares. These so-called *musharaka* and *mudaraba* contracts result in arrangements in which the profits are shared based on pre-agreed ratios while the liability for losses is in proportion to the individual capital contributions to the project (Hussain et al., 2015). In a *musharaka* transaction, both the entrepreneur and the investor provide capital to finance a project or investment and share ownership based on their respective capital shares. Both parties are entitled

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[1] Islamic financial contracts should adhere to the principles underlying Islamic banking and finance. As such, financial transactions involving activities such as charging interest, speculation, and short selling are prohibited (Iqbal and Mirakhor, 2011).

to take part in the management process. The (net) profits generated from the project are shared between the entrepreneur and the investor based on pre-agreed ratios while their liability for any (net) losses is in proportion to their initial capital contributions (Jobst, 2007). A *mudaraba* contract is similar to *musharaka*, but only the investor provides financing for the project, and only the entrepreneur can participate in management. The profits and losses are shared in the same way as with *musharaka* financing, and only the investor is liable for losses unless the manager is at fault (Jobst, 2007). Therefore, the main difference between a *musharaka* and a *mudaraba* transaction is in the level of financing provided by the investor.

In this paper, in addition to debt and equity financing, a nonlinear profit-and-loss-sharing arrangement with a fixed payment component is considered. This component can be either a wage for the entrepreneur or compensation for the investor. The structure of such an arrangement is inspired by Islamic profit-and-loss-sharing contracts. Similar to a *musharaka* or *mudaraba* contract, the nonlinear PLS arrangement explored in this paper involves the sharing of profits based on pre-agreed ratios and losses in proportion to initial capital contributions. However, the profits and losses are defined differently. While Islamic PLS contracts involve the sharing of net profits and losses resulting from a project, the nonlinear PLS contract considered here entails the sharing of the future value of the project, in other words, gross profits and losses. Another difference from a *musharaka* or *mudaraba* contract is the fixed payment component that can be agreed upon between the parties. These differences are illustrated with a simple numerical example in Appendix A.

The financial contract between the entrepreneur and the financier determines how the outcome, namely, profits and losses from the project, is shared between the parties to the contract. Conventional equity financing results in a linear profit-and-loss-sharing contract in which the share of the profits the financier is entitled to is the same as the share of the losses they are liable for. The profits and losses in this context refer to the increase and decrease in the market value of the project. Three alternative forms of financing are compared: (1) (risk-free) debt financing, (2) equity financing, and (3) a nonlinear profit-and-loss-sharing (PLS) contract with a fixed payment component. In essence, this is a form of hybrid financing that involves outcome-dependent profit and loss sharing, along with a fixed payment. The nonlinear profit-and-loss-sharing arrangement in this paper is unique in that the nonlinearity arises not only from the fixed payment component, but also from the fact that profits and losses are shared according to different ratios. Such a contract can be arranged by slightly modifying Islamic PLS contracts.

In the agency-theoretic framework applied in this paper, it is shown that both (risk-free) debt financing and the nonlinear PLS contract result in the first-best effort of the entrepreneur when both the entrepreneur and the financier are risk-neutral. Financing with equity capital does not offer an advantage in terms of risk sharing for the risk-neutral entrepreneur and dilutes their incentive to extend effort. Since risk-free debt financing is not always possible due to the non-negligible risk involved in starting a business, nonlinear PLS contracting is the superior method of financing.

The rest of this paper is structured as follows. A review of selected literature on principal-agent

relationships is provided in Section 2 to address issues regarding risk sharing, diminished managerial incentives, and the use of monitoring in such relationships. Section 3 outlines the theoretical model, detailing the underlying assumptions and specifying the alternative contracts under consideration. Section 4 presents theoretical analyses and results assuming that the entrepreneur is risk-neutral. It is demonstrated that a nonlinear incentive scheme can be designed so that even though the entrepreneur must share the outcome of their business with the outside investor, their incentive to exert effort is not diminished. Section 5 offers concluding remarks.

Moral Hazard Problem in Principal-Agent Relationships
Both financing with equity and a nonlinear profit-and-loss sharing arrangement, as considered in this paper, result in a partnership that involves risk sharing. Such a partnership is a principal-agent relationship between the entrepreneur (the agent) attempting to start a (small or medium-sized) business and the outside financier (the principal) who provides capital. A moral hazard problem arises when the agent can influence the outcome with their actions (i.e., effort), which the principal cannot observe. In this case, the Pareto-optimal level of risk sharing cannot be achieved since it would dilute the managerial incentives of the entrepreneur to exert a sufficient level of effort. On one hand, risk sharing is beneficial for the (risk-averse) entrepreneur; however, they would also have to share the outcome with the principal. Therefore, the informational asymmetries in principal-agent relationships result in a trade-off between the benefits of risk sharing and the diminished managerial incentives. This trade-off was first explored by Zeckhauser (1970) in the context of medical insurance. In this setting, optimal risk sharing can induce excessive spending (e.g., unnecessary visits to the emergency room) by insured individuals as they do not have to pay the full amount of their medical expenses. Therefore, the optimal insurance plan has to trade off risk sharing and incentives. In classical agency-theoretic literature, this trade-off in principal-agent relationships has been extensively inspected.

Ross (1973) explores the principal's problem, which is to find an optimal fee schedule that maximizes his von Neumann-Morgenstern utility while the agent chooses an action to maximize their utility. He finds that the solution to the principal's problem cannot be Pareto-optimal, but also claims that Pareto-efficiency can be achieved if the principal were aware of the action the agent would choose based on the fee schedule; then, they could simply require the agent to act in a certain way. However, this would also require monitoring of their actions. Holmström (1979) similarly suggests that the principal can force the agent to act a certain way if complete monitoring is possible. Even if the complete observation of the agent's effort is either impossible or too costly, any amount of information about their actions can alleviate the moral hazard problem. According to Shavell (1979), this information would only be valuable if the agent is risk-averse because a risk-neutral agent will absorb the risk and choose a level of effort to maximize the expected outcome, as their payment will depend only on this outcome. If the agent is risk-averse, their fee will also depend on the principal's knowledge about their actions. Holmström (1979) also considers that the principal has to bear the costs of monitoring; the value of the information about the agent's action depends on how costly it is to obtain. He suggests that the outcome must be

sufficiently bad in order for the benefits of monitoring to outweigh the costs. Spence and Zeckhauser (1971) examine monitoring costs in the context of medical insurance. The cost of information (i.e., monitoring) is also considered by Liu (1982) and Jost (1991).

Harris and Raviv (1979) conclude that it is optimal to combine imperfect monitoring with a nonlinear payment scheme; that is, if the principal decides to (imperfectly) monitor the agent's action, the optimal payment scheme to the agent depends on what the principal discovers regarding the agent's action. If the agent's action is found to be acceptable, they are paid based on a pre-determined outcome-dependent schedule. If not, they receive a lower fixed payment. Stiglitz (1974) also considers contracts between a landlord (principal) and a worker (agent) that consist of both a basic wage (i.e., a fixed payment) and an incentive payment in a simple agricultural economy. In his model, the incentive payment is linear in output. Spremann (1988) explores a principal-agent relationship resulting from such a profit-sharing arrangement that even though the principal can observe only the outcome (but not the agent's effort), they can choose an incentive scheme that induces the agent to (selfishly) decide for a level of effort that is acceptable for the principal. The agent can influence the outcome of the project by exerting effort, which also incurs private costs for the agent. In such a setting, the contract between the entrepreneur and the financier determines how the risk is allocated between them and to what extent the entrepreneur's incentives are diminished as a result of having to share the outcome with an outside financier.

By assuming that the agent and principal are both risk-neutral, and that the agent can influence the probability of "good" and "bad" outcomes, Breuer (1995) shows that nonlinear incentive contracts, where the agent receives an outcome-dependent and a fixed payment when the outcome is above a certain threshold and no payment at all otherwise, are optimal. The model employed by Breuer (1995) is similar to the one presented in this paper, where the agent has an influence not only on the expected outcome but also on the risk involved as the agent's actions affect not the expected outcome directly, but rather the probability of positive and negative outcomes. This approach differs from the LEN (Linear-Exponential-Normal) model proposed by Spremann (1988) in which the agent can only affect the expected outcome by shifting its probability distribution to the right.

In addition to debt and equity financing, this paper also considers nonlinear profit and loss sharing. In an agency-theoretic framework, nonlinear incentive schemes are hardly explored. Even in cases involving nonlinear incentive payments, the nonlinearity arises solely from the fixed payment element. In the setting explored in this paper, however, the nonlinearity results not only from the fixed payment component, but also because the profits and losses are shared based on different ratios. The motivation for considering such an intricate arrangement stems from the structure of Islamic PLS contracts. The nonlinear profit-and-loss-sharing contracting is compared to conventional methods of debt and equity financing with respect to the diminished managerial incentives to extend effort in order to find out if such an incentive scheme can alleviate the moral hazard problem in a principal-agent relationship when both the principal and the agent are risk-neutral, where risk sharing does not entail any benefits for the entrepreneur and does not play a role in their effort decision.

The Model

A one-period model of an investment opportunity for the entrepreneur, which requires an initial capital in the amount of A_0 in $t = 0$, is considered. The future value of this project in $t = 1$ is defined as $\tilde{A}_1 = (1 + \tilde{r}) A_0$ with $\tilde{r}$ as the uncertain rate of return on the investment. A profit on the investment is realized when the value of the project increases, that is, when the rate of return is positive. A negative rate of return results in a decrease in the value of the project and therefore, in a loss.

Assumption 1

The rate of return on the investment is defined as

$$\tilde{r} = y|\tilde{x}|, \tag{1}$$

where $\tilde{x}$ is a normally distributed random variable, $\tilde{x} \sim N(0, \sigma^2)$, and $y = \pm 1$ is an indicator of a positive or negative rate of return.

Assumption 2

The entrepreneur can increase the probability of a positive return (simultaneously decreasing the probability of a loss) with their effort, $e \in (0;1)$. The two-point distribution of y is then defined by the probability mass function

$$f(y;e) = \begin{cases} \dfrac{1}{2}(1-e) & \text{if} \quad y = -1 \\ \dfrac{1}{2}(1+e) & \text{if} \quad y = 1 \end{cases}. \tag{2}$$

The entrepreneur's effort is a decision variable and a measure of probability and results in private costs, $c(e)$, for the entrepreneur.

Assumption 3

The entrepreneur's cost of extending effort to facilitate the success of their business is

$$c(e) = \frac{1}{2}e^2 A_0, \tag{3}$$

which also depends on the initial capital A_0 required since the bigger the project, the more it costs (in monetary terms) for the entrepreneur to extend the effort.

Since the future value of the project is defined as $\tilde{A}_1 = (1 + \tilde{r}) A_0 = (1 + y|\tilde{x}|) A_0$, its probability distribution can be represented by the probability density function of $\tilde{r}$. The rate of return $\tilde{r}$ equals $-|\tilde{x}|$ with $\frac{1}{2}(1-e)$ probability and $|\tilde{x}|$ with $\frac{1}{2}(1+e)$ probability. Both $-|\tilde{x}|$ and $|\tilde{x}|$ have the same

probability density function.[2] Consequently, the probability density function[3] of $\tilde{r}$ can be expressed as

$$f(\tilde{r}) = \begin{cases} \dfrac{1-e}{\sigma\sqrt{2\pi}}\exp\left[-\dfrac{\tilde{r}^2}{2\sigma^2}\right] & \text{if} \quad \tilde{r} < 0 \\ \dfrac{1+e}{\sigma\sqrt{2\pi}}\exp\left[-\dfrac{\tilde{r}^2}{2\sigma^2}\right] & \text{if} \quad \tilde{r} > 0 \end{cases} = \dfrac{1+e\dfrac{\tilde{r}}{|\tilde{r}|}}{\sigma\sqrt{2\pi}}\exp\left[-\dfrac{\tilde{r}^2}{2\sigma^2}\right]. \tag{4}$$

Therefore, the entrepreneur can, through effort, influence both the expected value and the standard deviation of the project's rate of return.

The probability density function of the rate of return $\tilde{r}$ is illustrated in the figures below. Figure 1 represents the normal distribution of $\tilde{r}$ in the absence of effort of the entrepreneur. The probability of both a positive and a negative rate of return is $\dfrac{1}{2}$ without effort. The entrepreneur's effort influences these probabilities. The probability of a negative return is reduced to $\dfrac{1}{2}(1-e)$ while the probability of a positive return is increased to $\dfrac{1}{2}(1+e)$ when the entrepreneur extends effort. Figure 2 shows how the entrepreneur's effort affects (i.e., changes) the distribution of the rate of return.

Figure 1
*Probability Density Function $f(\tilde{r})$, $e = 0$*

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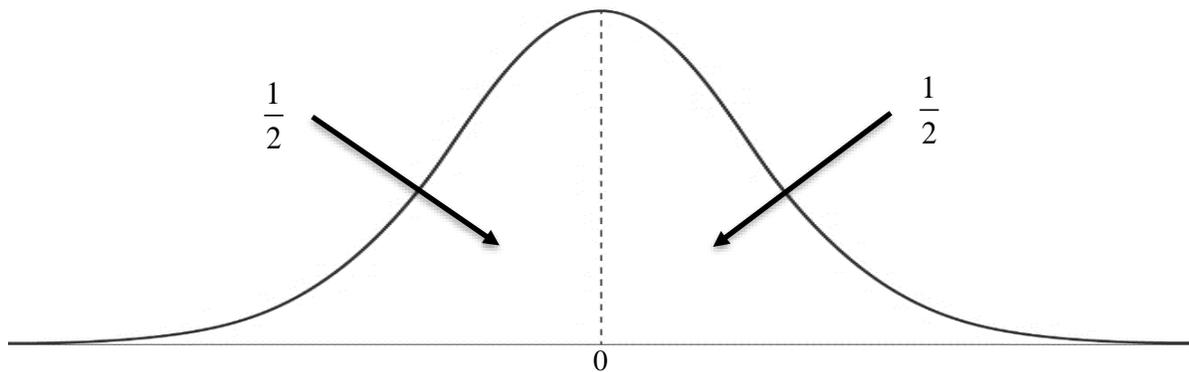
[2] $|\tilde{x}|$ and $-|\tilde{x}|$ exhibit a half-normal distribution, i.e. a folded normal distribution with 0 mean and the probability density function $f(|\tilde{x}|) = f(-|\tilde{x}|) = \dfrac{2}{\sigma\sqrt{2\pi}}\exp\left[-\dfrac{|\tilde{x}|^2}{2\sigma^2}\right]$ (Tsagris et al., 2014).

[3] The probability distribution of the rate of return $\tilde{r}$, whose properties are detailed in Appendix B, resembles a two-piece normal distribution (Wallis, 2014).

Figure 2

Probability Density Function $f(\tilde{r})$, $e > 0$
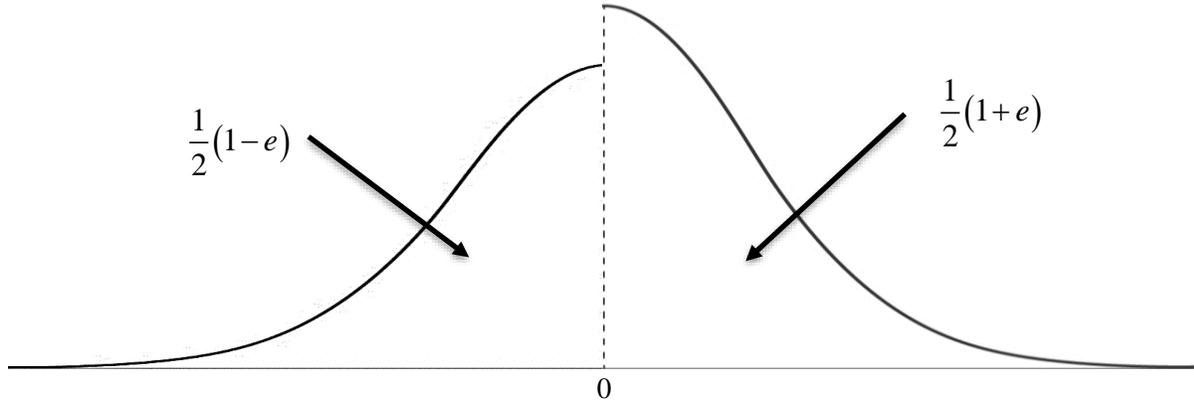
Assumption 4

The initial wealth $W_{0,E}$ of the entrepreneur E in $t = 0$ is not sufficient to cover the investment amount A_0; therefore, outside capital $C = A_0 - W_{0,E}$ is required for the entrepreneur to be able to start their business. They can either decide to invest their available capital in risk-free assets on the market instead of starting their business or raise the required capital as debt, equity, or with a nonlinear PLS contract.

Assumption 5

The interest rate charged on any debt is the same as the rate of return r_f on alternative (risk-free) investments. Debt financing is risk-free (free of default risk), and there are no market imperfections that would result in a difference between lending and borrowing rates.

     If the entrepreneur decides on a risk-free investment, their future wealth in $t = 1$ is $W_{1,E}^{risk-free\ investment} = (1 + r_f)W_{0,E}$. Alternatively, they can choose to finance the required capital with debt and start their business. In this case, they will have to exert effort as the manager and will have to repay the principal amount and the interest on the loan. If they can finance their business with risk-free debt capital, their uncertain future wealth will be

$$\tilde{W}_{1,E}^{debt} = \tilde{A}_1 - c(e) - (1 + r_f)C$$
$$= (1 + \tilde{r})A_0 - c(e) - (1 + r_f)(A_0 - W_{0,E}) \tag{5}$$

     Alternatively, the entrepreneur can finance their project with equity capital from a partner. The equity partner P provides capital in the amount of C_P and becomes a shareholder of the firm. This entitles him to a share α in the outcome of the project.

     In case of equity participation from an outside partner, the uncertain future wealth of the entrepreneur is

$$\tilde{W}_{1,E}^{equity} = (1-\alpha)\tilde{A}_1 - c(e) - (1+r_f)(C-C_P)$$
$$= (1-\alpha)(1+\tilde{r})A_0 - c(e) + (1+r_f)C_P - (1+r_f)(A_0 - W_{0,E}) \tag{6}$$

The amount of capital provided by the partner might be more or less than required. The entrepreneur can invest the difference at the risk-free rate r_f if $C-C_P < 0$. On the other hand, if $C-C_P > 0$, the entrepreneur will have to raise additional debt capital at the interest rate r_f.

The third option for the entrepreneur is a nonlinear profit-and-loss-sharing (PLS) contract with an investor. The nonlinear PLS investor I provides capital in the amount of $C_I = \lambda A_0$. The investor will then be entitled to a share γ of the outcome in case of a positive return (i.e., profit). The entrepreneur and the investor will share losses in proportion to their initial capital contributions, which means, the investor will be liable for a share λ of any losses. The entrepreneur and the investor will also agree on a fixed payment $B = \beta A_0$, which can be both positive and negative. When $B > 0$, the investor additionally receives a fixed compensation from the entrepreneur. Otherwise, i.e., if $B < 0$, the investor agrees to pay the entrepreneur a fixed wage on top of their share of the outcome.

As is the case with debt financing, the liability of the entrepreneur is not limited when they finance their business via a nonlinear PLS contract. The ramifications of limited liability are not considered in this paper.[4]

Assumption 6
In case the outcome of the entrepreneur's project is not sufficient for them to be able to fulfill their contractual obligations, they will (and can) cover the agreed-upon payments (principal and interest in case of any debt financing and the fixed payment to the investor (if $\beta > 0$) in case of financing with a nonlinear PLS contract) with their own private funds. Furthermore, even if the actual return on the project is too low, the investor will consider it bad luck rather than a result of the entrepreneur shirking their duties and still pay them the agreed-upon wage (if $\beta < 0$) with their own private funds (in case their share of the outcome is not sufficient to cover the fixed payment to the entrepreneur).

In the case of a nonlinear PLS arrangement with an investor, the uncertain future wealth of the entrepreneur is

$$\tilde{W}_{1,E}^{PLS} = \begin{cases} (1-\lambda)\tilde{A}_1 - B - c(e) - (1+r_f)(C-C_I) & \text{if} \quad \tilde{r} < 0 \\ (1-\gamma)\tilde{A}_1 - B - c(e) - (1+r_f)(C-C_I) & \text{if} \quad \tilde{r} > 0 \end{cases}$$
$$= \begin{cases} \left[(1-\lambda)(1+\tilde{r}) - \beta\right]A_0 - c(e) + (1+r_f)C_I - (1+r_f)(A_0 - W_{0,E}) & \text{if} \quad \tilde{r} < 0 \\ \left[(1-\gamma)(1+\tilde{r}) - \beta\right]A_0 - c(e) + (1+r_f)C_I - (1+r_f)(A_0 - W_{0,E}) & \text{if} \quad \tilde{r} > 0 \end{cases} \tag{7}$$

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[4] Innes (1990) explores the implications of limited liability (for both the risk-neutral principal and the risk-averse agent) for optimal effort and incentives when the agent chooses an effort before the state of nature is known.

The amount of capital provided by the investor might be more or less than required. If $C - C_I < 0$, the entrepreneur can invest the difference at the risk-free rate r_f. On the other hand, if $C - C_I > 0$, the entrepreneur will have to raise additional debt capital at the interest rate r_f.

The future wealth of both the entrepreneur and the potential financiers depends on the outcome of the investment and how that outcome is shared between them. The entrepreneur can increase the probability of a positive return and, consequently, the expected return on the investment with their effort, but that comes at a private cost (see (3)).

Assumption 7

The entrepreneur and the outside financiers are risk-neutral. The risk-neutral entrepreneur chooses a financial contract that maximizes the expected value of their uncertain future wealth, subsequently deciding on the level of effort. The entrepreneur's effort is known only to them. Potential risk-neutral financiers can observe only the outcome of the project without knowing the extent to which the entrepreneur's effort contributed to the outcome. The financiers, however, are not ignorant; they are aware of the relevant characteristics of the entrepreneur (attitude towards risk, cost of effort, etc.) and of the project they wish to finance. Therefore, the rational financiers can anticipate the effort decision of the rational entrepreneur.

Assumption 8

The outside financier is willing to provide such an amount of capital that their investment in the entrepreneur's project offers the same expected rate of return as their outside option (i.e., risk-free investment).

Comparison: Debt, Equity, and Nonlinear PLS

Three alternative financial contracts are considered: (i) risk-free debt financing, (ii) equity partnership with an outside partner P who provides capital C_P and is entitled to share α in the outcome of the entrepreneur's project, and (iii) nonlinear PLS contract with an investor I who provides capital $C_I = \lambda A_0$ and is entitled to share γ of the value of the project in case it increases while being liable for share λ of the losses. The nonlinear PLS contract also has a fixed component, meaning that either the entrepreneur or the investor receives a fixed payment (a wage or compensation, respectively).

Risk-free debt financing involving no profit and loss or risk sharing is considered first. If the entrepreneur finances their business with debt, the expected value of their future wealth from (5) is

$$
\begin{aligned}
E\left\{\tilde{W}_{1,E}^{debt}\right\} &= E\left\{\left(1+\tilde{r}\right)A_0 - c\left(e\right) - \left(1+r_f\right)\left(A_0 - W_{0,E}\right)\right\} \\
&= \left\{1 + e\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - \frac{1}{2}e^2\right\}A_0 - \left(1+r_f\right)\left(A_0 - W_{0,E}\right)
\end{aligned}
\tag{8}
$$

with the cost of effort function from (3). The entrepreneur then chooses the level of effort that

maximizes this expected value of their future wealth:

$$e = \frac{\sigma\sqrt{2}}{\sqrt{\pi}}. \tag{9}$$

This is the first-best level of effort since the entrepreneur does not share the outcome of the effort.

Assumption 9

To make sure Assumption 2, $e \in (0;1)$, holds, it is assumed that $\dfrac{\sigma\sqrt{2}}{\sqrt{\pi}} < 1 \iff \sigma < \sqrt{\dfrac{\pi}{2}}$.

With the first-best effort from (9), the entrepreneur's expected future wealth is

$$E\left\{\tilde{W}_{1,E}^{debt}\right\} = \left\{1 + \frac{\sigma^2}{\pi}\right\}A_0 - (1 + r_f)(A_0 - W_{0,E}). \tag{10}$$

They will start their business with debt financing only if it results in a higher expected future wealth than with a risk-free investment:

$$\left\{1 + \frac{\sigma^2}{\pi}\right\}A_0 - (1 + r_f)(A_0 - W_{0,E}) > (1 + r_f)W_{0,E} \iff \frac{\sigma^2}{\pi} > r_f. \tag{11}$$

That is, the expected rate of return, $\dfrac{\sigma^2}{\pi}$, on the investment, given the entrepreneur's effort from (9), should exceed the risk-free rate of return.

Financing with equity capital results in profit and loss and risk sharing. If the entrepreneur finances their business with equity participation from a partner, the expected value of their future wealth from (6) is

$$E\left\{\tilde{W}_{1,E}^{equity}\right\} = E\left\{(1-\alpha)(1+\tilde{r})A_0 - c(e) + (1+r_f)C_P - (1+r_f)(A_0 - W_{0,E})\right\}$$

$$= \left\{(1-\alpha)\left(1 + e\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right) - \frac{1}{2}e^2\right\}A_0 + (1+r_f)C_P - (1+r_f)(A_0 - W_{0,E}), \tag{12}$$

with the cost of effort function from (3). After the contract has been signed, that is, with given α and C_P, the entrepreneur chooses a level of effort that maximizes the expected value of their future wealth:

$$e = (1-\alpha)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}. \tag{13}$$

This is lower than the first-best effort from (9) since the entrepreneur has to share the benefits of their effort with the equity partner.

Lemma 1

The higher the share of the outcome the entrepreneur is to share with the outside partner, the lower the level of effort the entrepreneur is, *ceteris paribus*, willing to extend:

$$\frac{de}{d\alpha} = -\frac{\sigma\sqrt{2}}{\sqrt{\pi}} < 0 . \tag{14}$$

The participation constraint (which follows from Assumption 8) of the equity partner P with initial wealth $W_{0,P}$, who correctly anticipates the effort decision of the entrepreneur, determines the level of capital C_P they are willing to provide:

$$
\begin{aligned}
E\{\tilde{W}_{1,P}\} &= E\{\alpha(1+\tilde{r})A_0 + (1+r_f)(W_{0,P} - C_P)\} \\
&= \alpha\left\{1 + e\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\}A_0 + (1+r_f)(W_{0,P} - C_P) \\
&= (1+r_f)W_{0,P} \\
\Leftrightarrow\quad C_P &= \frac{1}{1+r_f}\alpha\left\{1 + e\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\}A_0
\end{aligned}
\tag{15}
$$

With the second-best effort from (13) and the amount of capital provided by the partner from (15), the expected value of the entrepreneur's future wealth is

$$E\{\tilde{W}_{1,E}^{equity}\} = \left\{1 + (1-\alpha^2)\frac{\sigma^2}{\pi}\right\}A_0 - (1+r_f)(A_0 - W_{0,E}) . \tag{16}$$

Considering $\alpha \in (0;1]$, financing with debt results in a higher expected future wealth for the entrepreneur than financing with equity (compare (10) and (160)) because there are no benefits from risk sharing for a risk-neutral entrepreneur, and the profit and loss sharing results in diluted managerial incentives to extend effort.

Another option for the entrepreneur to raise capital is a nonlinear PLS contract with an investor. If the entrepreneur finances their business via a nonlinear PLS arrangement with an investor, the expected value of their future wealth from (7) is

$$
\begin{aligned}
E\{\tilde{W}_{1,E}^{PLS}\} &= E\{[(1-\lambda)(1+\tilde{r}) - \beta]A_0 - c(e) + (1+r_f)C_I - (1+r_f)(A_0 - W_{0,E})\}\cdot\chi_{\{\tilde{r}<0\}} \\
&\quad + E\{[(1-\gamma)(1+\tilde{r}) - \beta]A_0 - c(e) + (1+r_f)C_I - (1+r_f)(A_0 - W_{0,E})\}\cdot\chi_{\{\tilde{r}>0\}} \\
&= \left\{\frac{1}{2}(1-e)(1-\lambda)\left\{1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} + \frac{1}{2}(1+e)(1-\gamma)\left\{1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} - \frac{1}{2}e^2 - \beta\right\}A_0 \\
&\quad + (1+r_f)C_I - (1+r_f)(A_0 - W_{0,E})
\end{aligned}
\tag{17}
$$

with the cost of effort function from (3). Here, $\chi_{\{\tilde{r}<0\}} = \begin{cases} 1 & \text{if} \quad \tilde{r} < 0 \\ 0 & \text{if} \quad \tilde{r} > 0 \end{cases}$ and $\chi_{\{\tilde{r}>0\}} = \begin{cases} 1 & \text{if} \quad \tilde{r} > 0 \\ 0 & \text{if} \quad \tilde{r} < 0 \end{cases}$

are indicator functions.

The entrepreneur then chooses a level of effort that maximizes this expected value of their future wealth after the contract has been signed, i.e., with given λ (and C_I), γ, and β:

$$e = \frac{\sigma\sqrt{2}}{\sqrt{\pi}} + \frac{1}{2}\lambda\left\{1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} - \frac{1}{2}\gamma\left\{1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\}. \tag{18}$$

Lemma 2

The higher (lower) the share of the negative (positive) outcome the investor is entitled to, the higher the level of effort the entrepreneur is, *ceteris paribus*, willing to extend:

$$\frac{de}{d\lambda} = \frac{1}{2}\left(1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right) > 0, \tag{19}$$

and

$$\frac{de}{d\gamma} = -\frac{1}{2}\left(1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right) < 0. \tag{20}$$

There are both positive and negative effects of the nonlinear profit-and-loss-sharing arrangement on the managerial incentives of the entrepreneur to extend effort. On one hand, being entitled to a lower share of the negative outcome (when λ increases) is a positive incentive for the entrepreneur; as λ increases, they extend a higher level of effort to decrease the probability of the negative outcome. On the other hand, the entrepreneur is less motivated to increase the probability of a positive outcome when they have to give up a higher share of it (when γ increases). Profit sharing can then be arranged by choice of optimal γ (dependent on λ) in such a way that the positive and negative effects on the incentives of the entrepreneur to extend effort cancel each other out:

$$\frac{de}{d\lambda} + \frac{de}{d\gamma} \cdot \frac{d\gamma}{d\lambda} = \frac{1}{2}\left(1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right) - \frac{1}{2}\left(1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right) \cdot \frac{d\gamma}{d\lambda} = 0 \quad \Leftrightarrow \quad \frac{d\gamma}{d\lambda} = \frac{1 - \dfrac{\sigma\sqrt{2}}{\sqrt{\pi}}}{1 + \dfrac{\sigma\sqrt{2}}{\sqrt{\pi}}}. \tag{21}$$

The participation constraint (which follows from Assumption 8) of the PLS investor I with initial wealth $W_{0,I}$, who correctly anticipates the effort decision of the entrepreneur, determines the fixed payment either they (if $B > 0$) or the entrepreneur (if $B < 0$) is going to receive:

$$E\{\tilde{W}_{1,I}\} = E\{[\lambda(1+\tilde{r})+\beta]A_0 + (1+r_f)(W_{0,I}-C_I)\} \cdot \chi_{\{\tilde{r}<0\}}$$

$$+ E\{[\gamma(1+\tilde{r})+\beta]A_0 + (1+r_f)(W_{0,I}-C_I)\} \cdot \chi_{\{\tilde{r}>0\}}$$

$$= \left[\frac{1}{2}(1-e)\lambda\left\{1-\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} + \frac{1}{2}(1+e)\gamma\left\{1+\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} + \beta\right]A_0 + (1+r_f)(W_{0,I}-C_I)$$

$$= \left[\frac{1}{2}(1-e)\lambda\left\{1-\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} + \frac{1}{2}(1+e)\gamma\left\{1+\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} + \beta\right]A_0 + (1+r_f)(W_{0,I}-\lambda A_0) \qquad (22)$$

$$= (1+r_f)W_{0,I}$$

$$\Leftrightarrow \quad B = \beta A_0 = \left[(1+r_f)\lambda - \frac{1}{2}(1-e)\lambda\left\{1-\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} - \frac{1}{2}(1+e)\gamma\left\{1+\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\}\right]A_0$$

Considering $C_I = \lambda A_0$, with the effort from (18) and the fixed payment from (22), the entrepreneur's expected future wealth is

$$E\{\tilde{W}_{1,E}^{PLS}\} = \left\{1 + \frac{\sigma^2}{\pi} - \frac{1}{8}\left[(\lambda+\gamma)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - (\lambda-\gamma)\right]^2\right\}A_0 - (1+r_f)(A_0 - W_{0,E}), \qquad (23)$$

to be maximized by the optimal share of the positive outcome the entrepreneur is to give up:

$$\frac{dE\{\tilde{W}_{1,E}^{PLS}\}}{d\gamma} = 0 \quad \Leftrightarrow \quad \gamma = \frac{1-\dfrac{\sigma\sqrt{2}}{\sqrt{\pi}}}{1+\dfrac{\sigma\sqrt{2}}{\sqrt{\pi}}}\lambda. \qquad (24)$$

With the optimal γ from (24), the entrepreneur's effort from (18) becomes

$$e = \frac{\sigma\sqrt{2}}{\sqrt{\pi}} + \frac{1}{2}\lambda\left\{1-\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} - \frac{1}{2}\frac{1-\dfrac{\sigma\sqrt{2}}{\sqrt{\pi}}}{1+\dfrac{\sigma\sqrt{2}}{\sqrt{\pi}}}\lambda\left\{1+\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} = \frac{\sigma\sqrt{2}}{\sqrt{\pi}}. \qquad (25)$$

This corresponds to the first-best level of effort (see (9)).

Lastly, the fixed payment $B = \beta A_0$ can be determined from (22) with the optimal effort from (18) (or (25)) and the optimal γ from (24):

$$\beta = \lambda\left(r_f + \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right) \quad \Leftrightarrow \quad B = \lambda\left(r_f + \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)A_0. \qquad (26)$$

With the first-best level of effort, the expected future wealth of the entrepreneur is also at the

same level as with debt financing:[5]

$$E\left\{\tilde{W}_{1,E}^{PLS}\right\} = \left\{1 + \frac{\sigma^2}{\pi}\right\} A_0 - \left(1 + r_f\right)\left(A_0 - W_{0,E}\right), \tag{27}$$

which means they will start their business via a nonlinear PLS transaction only if (see (11)).

Financing via a nonlinear PLS transaction also results in the first-best level of effort and the same level of expected future wealth for the risk-neutral entrepreneur as risk-free debt financing. Risk sharing with the PLS investor does not have any benefits for the risk-neutral entrepreneur. However, their incentives to extend effort are also not diluted; with the first-best effort and the optimal share of the positive outcome, the entrepreneur is willing to give up, and there is no disadvantage to sharing the outcome with an outside investor. Therefore, when risk-free debt is not readily available to the entrepreneur, they can finance their project through a nonlinear PLS transaction and achieve the same expected rate of return.

The nonlinear profit-and-loss sharing arrangement analyzed in this paper can be constructed using a combination of a slightly modified Islamic profit-and-loss-sharing *musharaka* (or *mudaraba*) contract, where the gross profits instead of net profits are shared between the parties based on a pre-agreed ratio, and a *murabaha*[6] transaction, which would be used to turn the fixed payment component into a markup sale to be paid off in installments.

The fixed payment (see (26)) from the nonlinear PLS transaction is positive, that is, the investor is to receive this payment. The total payment to the investor, including their share of the outcome of the project, can be constructed as a *murabaha* transaction, where the entrepreneur, essentially, buys out the investor at the end of the project at a markup. Then, instead of receiving their share of the outcome and the fixed payment immediately at the end of the project, the investor would agree to sell their share to the entrepreneur at a markup to be paid in installments over a period of time, which constitutes a *murabaha* sale. The only caveat is that the installment payment in a *murabaha* transaction has to be determined from the beginning, which is not possible since the outcome of the project is uncertain. However, the parties can agree beforehand on a markup payment that will have to be paid as part of every installment after the project is concluded. The present value of the expected installment payments has to then equal the initial capital contribution of the investor.

The expected value of the investor's share at the end of the project is

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[5] The positive term $\frac{1}{8}\left[\left(\lambda + \gamma\right)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} - \left(\lambda - \gamma\right)\right]^2$ in (23) represents the entrepreneur's disadvantage of sharing the outcome of her business with the investor. With the optimal γ from (24), this disadvantage is zero.

[6] A *murabaha* contract is a markup sale of assets. In such a transaction, the lender purchases an asset (or already owns it) and sells it to the borrower at a higher price to be paid in installments in the future (Jobst, 2007).

$$E\left\{\lambda\left(1+\tilde{r}\right)A_0\right\}\cdot\chi_{\{\tilde{r}<0\}}+E\left\{\gamma\left(1+\tilde{r}\right)A_0\right\}\cdot\chi_{\{\tilde{r}>0\}}$$

$$=\left[\frac{1}{2}\left(1-e\right)\lambda\left\{1-\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\}+\frac{1}{2}\left(1+e\right)\gamma\left\{1+\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\}\right]A_0=\lambda\left(1-\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)A_0 \tag{28}$$

with effort from (18) (or (25)) and the profit share γ from (24). Instead of receiving the fixed payment B from (26) immediately at the end of the project, the investor would agree to sell their share to the entrepreneur at a markup mA_0 to be paid in n installments after the project is concluded. Then, an expected installment payment would be $\dfrac{\lambda\left(1-\dfrac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)+m}{n}A_0$, which can be considered an annuity. The present value of this annuity should equal the initial capital contribution of the investor:

$$C_I=\lambda A_0=\frac{1}{1+r_f}\cdot\frac{\left(1+r_f\right)^n-1}{\left(1+r_f\right)^n r_f}\cdot\frac{\lambda\left(1-\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)+m}{n}A_0. \tag{29}$$

The markup from the *murabaha* transaction can then be determined as

$$mA_0=\lambda\left[\frac{n\left(1+r_f\right)^{n+1}r_f}{\left(1+r_f\right)^n-1}-\left(1-\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)\right]A_0. \tag{30}$$

Conclusion

The principal-agent relationship between a risk-neutral entrepreneur (the agent) and a risk-neutral financier (the principal) is analyzed for different financial contracts. The type of financial contract used has an impact on managerial incentives. Three alternatives for an entrepreneur seeking capital to finance their project have been explored: i) risk-free debt financing, ii) equity financing with capital provided by a partner, iii) nonlinear profit-and-loss-sharing contracting with an investor. The effects of sharing the outcome of the project with the equity partner and the nonlinear PLS investor on the optimal effort and expected future wealth of the risk-neutral entrepreneur have been analyzed. It is assumed that the entrepreneur can influence the outcome of the project with their effort, which also results in private costs for the entrepreneur. They can, through their effort, increase the probability of a profit, thereby decreasing the probability of a loss. The nonlinear PLS contracting considered in this paper involves sharing profits (an increase in the value of the project) and losses (a decrease in the value of the project) based on different ratios between the entrepreneur and the investor and a fixed payment that either the entrepreneur or the investor is to receive. In that case, the negative outcome is shared between the parties based on their initial capital contributions to the project. The entrepreneur and investor also agree on their shares of the positive outcome and the fixed payment. This transaction can be arranged by combining a slightly modified Islamic PLS contract (*musharaka* or *mudaraba*) and a *murabaha* sale.

Financing with (risk-free) debt capital does not involve any profit and loss or risk sharing. In

that case, the optimal effort of the entrepreneur is first-best since they do not have to share the outcome with an outside financier. However, risk-free debt financing is usually not possible. With equity or nonlinear PLS financing, the entrepreneur shares both the outcome and the risk with the outside financier. For the risk-neutral entrepreneur, risk sharing does not entail an advantage. With equity financing, the incentives of the entrepreneur to extend effort are diminished, resulting in a lower level of effort. Therefore, financing with equity capital is inferior to risk-free debt financing and results in a lower expected future wealth for the entrepreneur. The nonlinear PLS transaction, on the other hand, can be arranged in such a way that the incentives of the entrepreneur to extend effort are not diluted. In that case, the entrepreneur's effort is also first-best, and the nonlinear PLS contracting is an alternative to risk-free debt financing, that is, it results in the same expected future wealth of the entrepreneur. Considering the entrepreneur might not have access to risk-free debt capital, the nonlinear PLS arrangement is then optimal for the risk-neutral entrepreneur.

The analysis in this paper has demonstrated the advantages of a nonlinear profit-and-loss-sharing contract. Generally speaking, outcome-dependent payments to the agent in a principal-agent relationship can be structured to mitigate the dilution of the managerial incentives of the agent to extend the effort. An extension would combine the ramifications of risk sharing and diluted managerial incentives for a risk-averse entrepreneur. Financing with risk-free debt capital would again result in the first-best effort of the entrepreneur. For the risk-averse entrepreneur, however, this would not indicate that debt financing is optimal since the benefits of risk sharing with an outside financier have to be also considered. When financing with equity or via a nonlinear PLS arrangement, the optimal contract structure has to trade off the benefits of risk sharing and diminished managerial incentives of the entrepreneur to extend effort. Financing with equity will dilute the incentives of the entrepreneur and result in a second-best effort. It can, *a priori*, be claimed that the risk-averse entrepreneur's effort in the case of nonlinear PLS contracting will be lower than the first-best effort but higher than the second-best effort since it is a hybrid financial instrument combining elements of both debt and equity financing. Which financing alternative is more advantageous for the risk-averse entrepreneur will then depend on the trade-off between the benefits of risk sharing and the adverse effects of the moral hazard problem resulting from financing with equity or via the nonlinear PLS contract.

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Appendix A
Difference between a *musharaka* contract and the nonlinear PLS

The nonlinear PLS arrangement considered in this paper was inspired by profit-and-loss-sharing *musharaka* contracts in Islamic finance; however, there are differences between them. Based on a *musharaka* contract, the net losses and profits are shared based on (possibly) different shares. According to the nonlinear PLS contract considered here, on the other hand, the different profit and loss shares apply to the future value of the project. In essence, gross profits and losses are shared based on different ratios. This arrangement also entails a fixed payment to one of the parties to the contract, which does not apply to *musharaka* financing. These differences are illustrated below with a simple numerical example.

Suppose that the entrepreneur has an investment opportunity that requires an initial investment amount of 100 in $t = 0$. The future value of the project can be either 80 (20% a decrease in value, i.e., loss) or 120 (20% an increase in value, i.e., profit) in $t = 1$. If the entrepreneur finances the project with an investor through a *musharaka* contract, where the investor provides 60% of the capital, the entrepreneur will be liable for only 40% of the losses. Assuming the parties agree to share the net profits fifty-fifty, the cash flows and the returns for the investor and entrepreneur are calculated as follows:

Table 1
Example: Cash Flows and Returns Resulting from a Musharaka Contract

Investment		Investor				Entrepreneur		
$t = 0$	$t = 1$	$t = 0$	$t = 1$			$t = 0$	$t = 1$	
-100	80	-60	$60 - 0.60 \cdot 20 = 48$	-20.00%		-40	$40 - 0.40 \cdot 20 = 32$	-20.00%
	120		$60 + 0.50 \cdot 20 = 70$	16.67%			$40 + 0.50 \cdot 20 = 50$	25.00%

The cash flow to the investor in case the value of the project in $t = 1$ turns out to be 80 is their initial capital contribution less 60% of the decrease in value. Their rate of return is then calculated as $\frac{60 - 0.60 \cdot 20}{60} - 1 = \frac{48}{60} - 1 = -0.20$. The entrepreneur, on the other hand, is liable for 40% of the decrease in the value of the project. Consequently, her rate of return will be $\frac{40 - 0.40 \cdot 20}{40} - 1 = \frac{32}{40} - 1 = -0.20$, which is the same rate of return as the investor's since the losses are shared in proportion to initial capital contributions. When the value of the project increases and turns out to be 120 in $t = 1$, the investor is entitled to 50% of this increase based on the *musharaka* contract. Therefore, their rate of return will be $\frac{60 + 0.50 \cdot 20}{60} - 1 = \frac{70}{60} - 1 \approx 0.17$. The entrepreneur is also entitled to 50% of the 20 increase in the value of the project, resulting in a rate of return higher than the investor's: $\frac{40 + 0.50 \cdot 20}{40} - 1 = \frac{50}{40} - 1 = 0.25$.

Based on the nonlinear PLS arrangement considered in this paper, the future value of the project, rather than net profits (or losses) from it, is shared between the parties to the contract, which also includes a fixed payment. If the investor provides 60% of the capital, they are liable for 60% of the losses. Assuming that the investor agrees to the 50% share of the value of the project in case the value increases, and that they are also promised a fixed payment amounting to 5% of the initial investment amount, the cash flows and returns for the investor and the entrepreneur can be calculated as follows:

Table 2

Example: Cash Flows and Returns Resulting from the Nonlinear PLS Contract

Investment		Investor			Entrepreneur		
$t=0$	$t=1$	$t=0$	$t=1$		$t=0$	$t=1$	
-100	80	-60	$0.60\cdot80+5=53$	-11.67%	-40	$0.40\cdot80-5=27$	-32.50%
	120		$0.50\cdot120+5=65$	8.33%		$0.50\cdot120-5=55$	37.50%

As agreed upon in the nonlinear PLS contract, in case the value of the project decreases, the investor is entitled to 60% of the value of the project in $t=1$ and a fixed payment that corresponds to 5% of the initial investment amount 100. Their rate of return from the investment is then $\dfrac{0.60\cdot80+0.05\cdot100}{60}-1=\dfrac{53}{60}-1\approx-0.12$. The entrepreneur, on the other hand, is entitled to 40% of the value of the project and has to provide the agreed upon fixed payment to the investor, resulting in a rate of return of $\dfrac{0.40\cdot80-0.05\cdot100}{40}-1=\dfrac{27}{40}-1\approx-0.33$. If the project is successful and is valued at 120 in $t=1$, only 50% of it belongs to the investor who also receives the fixed payment (5% of the initial investment amount 100). The rate of return is then $\dfrac{0.50\cdot120+0.05\cdot100}{60}-1=\dfrac{65}{60}-1\approx0.08$ for the investor. The entrepreneur's share of the value of the project, in this case, is also 50%, and they also have to pay the fixed amount to the investor. Her rate of return can then be calculated as $\dfrac{0.50\cdot120-0.05\cdot100}{40}-1=\dfrac{55}{40}-1\approx0.38$.

When the entrepreneur finances the project by entering into a *musharaka* contract with an investor, the rate of return for both parties is the same as the rate of return on the project in case the value of the project decreases since they share net losses in accordance with their initial capital contributions. The sharing rule for the net profits can, however, be arranged so that the entrepreneur receives a higher return than the investor in case the value of the project increases since they are the ones managing the project. Here, the entrepreneur gets rewarded when the outcome is good, but does not get punished for a bad outcome. In a nonlinear PLS arrangement considered in this paper, however, the rules for sharing profits and losses are slightly different than in a *musharaka* contract, and there is also a fixed payment component. The simple numerical example presented here shows that the rate of return for the investor and the entrepreneur are higher and lower (compared to *musharaka*), respectively, if the value of the project decreases. In essence, the entrepreneur is punished for the bad outcome. On the other hand, the investor and the entrepreneur receive lower and higher returns (compared to *musharaka*), respectively, when the project value increases; the entrepreneur is, in a way, rewarded for the good outcome.

Appendix B
Effort-dependent two-piece normal distribution of the rate of return

The uncertain rate of return on the investment is defined by the probability density function

$$f(\tilde{r}) = \frac{1 + e\frac{\tilde{r}}{|\tilde{r}|}}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right]. \qquad \text{see (4)}$$

The probabilities of a negative and a positive return (i.e., the probabilities of losses and profits) are then

$$\Pr(\tilde{r} < 0) = \int_{-\infty}^{0} f(\tilde{r})d\tilde{r} = \frac{1}{2}(1-e), \qquad (31)$$

and

$$\Pr(\tilde{r} > 0) = \int_{0}^{+\infty} f(\tilde{r})d\tilde{r} = \frac{1}{2}(1+e), \qquad (32)$$

respectively. The expected negative return, positive return, and total return are derived using the probability density function (4):

$$E(\tilde{r}) \cdot 1_{\{\tilde{r}<0\}} = \Pr(\tilde{r}<0) \cdot E(\tilde{r} \mid \tilde{r}<0)$$
$$= \int_{-\infty}^{0} \tilde{r}f(\tilde{r})d\tilde{r} = -\frac{1}{2}(1-e)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}, \qquad (33)$$

$$E(\tilde{r}) \cdot 1_{\{\tilde{r}>0\}} = \Pr(\tilde{r}>0) \cdot E(\tilde{r} \mid \tilde{r}>0)$$
$$= \int_{0}^{+\infty} \tilde{r}f(\tilde{r})d\tilde{r} = \frac{1}{2}(1+e)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}, \qquad (34)$$

and

$$E(\tilde{r}) = E(\tilde{r}) \cdot \chi_{\{\tilde{r}<0\}} + E(\tilde{r}) \cdot \chi_{\{\tilde{r}>0\}}$$
$$= -\frac{1}{2}(1-e)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} + \frac{1}{2}(1+e)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} = e\frac{\sigma\sqrt{2}}{\sqrt{\pi}}. \qquad (35)$$

The corresponding risks, that is, the lower semi-variance, upper semi-variance, and total variance are

$$Var(\tilde{r}) \cdot \chi_{\{\tilde{r}<0\}} = E(\tilde{r}^2) \cdot \chi_{\{\tilde{r}<0\}} - \left[E(\tilde{r}) \cdot \chi_{\{\tilde{r}<0\}}\right]^2$$
$$= \Pr(\tilde{r}<0) \cdot E(\tilde{r}^2 \mid \tilde{r}<0) - \Pr(\tilde{r}<0) \cdot \left[E(\tilde{r} \mid \tilde{r}<0)\right]^2$$
$$= \int_{-\infty}^{0} \tilde{r}^2 f(\tilde{r})d\tilde{r} - \frac{\left[\int_{-\infty}^{0} \tilde{r}f(\tilde{r})d\tilde{r}\right]^2}{\Pr(\tilde{r}<0)}, \qquad (36)$$
$$= \frac{1}{2}(1-e)\sigma^2 - \frac{\left[-\frac{1}{2}(1-e)\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right]^2}{\frac{1}{2}(1-e)} = \frac{1}{2}(1-e)\sigma^2\left(1-\frac{2}{\pi}\right)$$

$$Var(\tilde{r}) \cdot \chi_{\{\tilde{r}>0\}} = E(\tilde{r}^2) \cdot \chi_{\{\tilde{r}>0\}} - \left[E(\tilde{r}) \cdot \chi_{\{\tilde{r}>0\}} \right]^2$$

$$= \Pr(\tilde{r}>0) \cdot E(\tilde{r}^2 \mid \tilde{r}>0) - \Pr(\tilde{r}>0) \cdot \left[E(\tilde{r} \mid \tilde{r}>0) \right]^2$$

$$= \int_0^{+\infty} \tilde{r}^2 f(\tilde{r}) d\tilde{r} - \frac{\left[\int_0^{+\infty} \tilde{r} f(\tilde{r}) d\tilde{r} \right]^2}{\Pr(\tilde{r}>0)} \qquad , \qquad (37)$$

$$= \frac{1}{2}(1+e)\sigma^2 - \frac{\left[\frac{1}{2}(1+e)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]^2}{\frac{1}{2}(1+e)} = \frac{1}{2}(1+e)\sigma^2 \left(1 - \frac{2}{\pi} \right)$$

and

$$Var(\tilde{r}) = E(\tilde{r}^2) - \left[E(\tilde{r}) \right]^2$$

$$= \left\{ E(\tilde{r}^2) \cdot \chi_{\{\tilde{r}<0\}} + E(\tilde{r}^2) \cdot \chi_{\{\tilde{r}>0\}} \right\} - \left[E(\tilde{r}) \cdot \chi_{\{\tilde{r}<0\}} + E(\tilde{r}) \cdot \chi_{\{\tilde{r}>0\}} \right]^2$$

$$= \left\{ \frac{1}{2}(1-e)\sigma^2 + \frac{1}{2}(1+e)\sigma^2 \right\} - \left[-\frac{1}{2}(1-e)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} + \frac{1}{2}(1+e)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]^2 \qquad (38)$$

$$= \sigma^2 - \left[e\frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]^2 = \sigma^2 \left(1 - \frac{2e^2}{\pi} \right)$$

The entrepreneur's optimal effort increases in standard deviation (see (9) as reference) for two main reasons:

I.   As standard deviation σ increases, the expected negative return decreases while the expected positive return and expected total return increase for a given level of effort $e \in (0;1)$. An increase in effort can then limit the decrease in the expected negative return while boosting the increase in the expected positive return and expected total return resulting from the increase in standard deviation since an increase in effort leads to an increase in expected negative return, expected positive return, and expected total return:

$$\frac{d^2 E(\tilde{r}) \cdot \chi_{\{\tilde{r}<0\}}}{d\sigma de} = \frac{d\left\{ \frac{dE(\tilde{r}) \cdot \chi_{\{\tilde{r}<0\}}}{d\sigma} \right\}}{de} = \frac{d\left\{ \frac{d\left[-\frac{1}{2}(1-e)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]}{d\sigma} \right\}}{de} , \qquad (39)$$

$$= \frac{d\left\{ -\frac{1-e}{\sqrt{2\pi}} \right\}}{de} = \frac{1}{\sqrt{2\pi}} > 0$$

$$\frac{d^2 E(\tilde{r}) \cdot \chi_{\{\tilde{r}>0\}}}{d\sigma de} = \frac{d\left\{ \frac{dE(\tilde{r}) \cdot \chi_{\{\tilde{r}>0\}}}{d\sigma} \right\}}{de} = \frac{d\left\{ \frac{d\left[\frac{1}{2}(1+e)\frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]}{d\sigma} \right\}}{de}$$

$$= \frac{d\left\{ \frac{1+e}{\sqrt{2\pi}} \right\}}{de} = \frac{1}{\sqrt{2\pi}} > 0 \qquad , \qquad (40)$$

and

$$\frac{d^2 E(\tilde{r})}{d\sigma de} = \frac{d\left\{\dfrac{dE(\tilde{r})}{d\sigma}\right\}}{de} = \frac{d\left\{\dfrac{d\left[e\dfrac{\sigma\sqrt{2}}{\sqrt{\pi}}\right]}{d\sigma}\right\}}{de}.$$

(41)

$$= \frac{d\left\{e\dfrac{\sqrt{2}}{\sqrt{\pi}}\right\}}{de} = \frac{2}{\sqrt{2\pi}} > 0$$

II.   As the standard deviation σ increases, the lower semi-variance, the upper semi-variance, and the total variance increase for a given level of effort $e \in (0;1)$. An increase in effort can then limit the increase in the lower semi-variance and total variance while boosting the increase in the upper semi-variance, resulting from the increase in standard deviation, since an increase in effort leads to an increase in the upper semi-variance and a decrease in the lower semi-variance and total variance:

$$\frac{d^2 Var(\tilde{r})\cdot\chi_{\{\tilde{r}<0\}}}{d\sigma de} = \frac{d\left\{\dfrac{dVar(\tilde{r})\cdot\chi_{\{\tilde{r}<0\}}}{d\sigma}\right\}}{de} = \frac{d\left\{\dfrac{d\left[\dfrac{1}{2}(1-e)\sigma^2\left(1-\dfrac{2}{\pi}\right)\right]}{d\sigma}\right\}}{de},$$

(42)

$$= \frac{d\left\{(1-e)\sigma\left(1-\dfrac{2}{\pi}\right)\right\}}{de} = -\sigma\left(1-\dfrac{2}{\pi}\right) < 0$$

$$\frac{d^2 Var(\tilde{r})\cdot\chi_{\{\tilde{r}>0\}}}{d\sigma de} = \frac{d\left\{\dfrac{dVar(\tilde{r})\cdot\chi_{\{\tilde{r}>0\}}}{d\sigma}\right\}}{de} = \frac{d\left\{\dfrac{d\left[\dfrac{1}{2}(1+e)\sigma^2\left(1-\dfrac{2}{\pi}\right)\right]}{d\sigma}\right\}}{de},$$

(43)

$$= \frac{d\left\{(1+e)\sigma\left(1-\dfrac{2}{\pi}\right)\right\}}{de} = \sigma\left(1-\dfrac{2}{\pi}\right) > 0$$

and

$$\frac{d^2 Var(\tilde{r})}{d\sigma de} = \frac{d\left\{\dfrac{dVar(\tilde{r})}{d\sigma}\right\}}{de} = \frac{d\left\{\dfrac{d\left[\sigma^2\left(1-\dfrac{2e^2}{\pi}\right)\right]}{d\sigma}\right\}}{de}.$$

(44)

$$= \frac{d\left\{2\sigma\left(1-\dfrac{2e^2}{\pi}\right)\right\}}{de} = -8e\frac{\sigma}{\pi} < 0.$$

Conflict of Interests
No, there are no conflicting interests.